4.7. Euler - Cauchy Equation

Consider $\underline{a}x^2 y'' + \underline{b}x y' + \underline{c}y = 0, \; x>0$ (1)

Sub → Transform the equation to one with constant coefficients

$$x = e^t, \quad t = \ln x, \quad \frac{dx}{dt} = e^t = x, \quad \frac{dt}{dx} = \frac{1}{x}$$

$$\dot{y} = \frac{dy}{dt}, \quad y' = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx} \Rightarrow y' = \frac{1}{x}\dot{y} = \frac{\dot{y}}{x}$$

$$\frac{d^2y}{d^2x} = \frac{d}{dx}\left(\frac{\dot{y}}{x}\right) = -\frac{1}{x^2}\dot{y} + \frac{1}{x}\left(\frac{d}{dx}\,\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\dot{y} + \frac{1}{x}\,\frac{1}{x}\,\frac{d}{dt}\,\frac{dy}{dt}$$

$$= -\frac{1}{x^2}\dot{y} + \frac{1}{x^2}\,\frac{d^2y}{dt^2} = -\frac{1}{x^2}\dot{y} + \frac{1}{x^2}\ddot{y}$$

Rewrite (1) in terms of the new variable t:

$$ax^2\underbrace{\left(-\frac{1}{x^2}\dot{y} + \frac{1}{x^2}\ddot{y}\right)}_{y'' = \frac{d^2y}{d^2x}} + bx\underbrace{\left(\frac{\dot{y}}{x}\right)}_{y'} + cy = 0$$

$$a(-\dot{y}+\ddot{y}) + b\dot{y} + cy = 0$$

$$\longleftrightarrow \underbrace{a\ddot{y} + (b-a)\dot{y} + cy = 0}_{\text{Linear eq with constant coefficients}}$$

Char polynomial: $\boxed{ar^2 + (b-a)r + c = 0}$

$\Delta = (b-a)^2 - 4ac.$

$\Delta > 0$, 2 real roots $r_1 \neq r_2$,

$$y = c_1 e^{tr_1} + c_2 e^{tr_2}, \quad t = \ln x$$

$$y = c_1 e^{(\ln x) r_1} + c_2 e^{(\ln x) r_2} = c_1 x^{r_1} + c_2 x^{r_2}$$

$\Delta = 0$, Double roots $r_1 = r_2 = r$

$$y = c_1 e^{tr_1} + c_2 t e^{tr_1} = c_1 x^{r_1} + c_2 (\ln x) x^{r_1}$$

$\Delta < 0$, 2 complex roots $\alpha \pm i\beta$

$$y = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)$$
$$= c_1 x^{\alpha} \cos(\beta \ln x) + c_2 x^{\alpha} \sin(\beta \ln x).$$

Ex: $x^2 y'' - 2y = 0$

$a = 1, \; b = 0, \; c = -2.$

Char polynomial: $ar^2 + (b-a)r + c = 0$

$\iff r^2 - 1r - 2 = 0$

$\iff r = -1, \; r = 2$

$y = c_1 x^{-1} + c_2 x^2$

Ex: $x^2 y'' + xy' + 4y = 0$

$a = 1, \; b = 1, \; c = 4$

Char polynomial: $r^2 + 4 = 0$

$\iff$ complex roots $r = 0 \pm 2i$, $\alpha = 0, \; \beta = 2.$

$$y = c_1 x^0 \cos(2\ln(x)) + c_2 x^0 \sin(2\ln(x))$$
$$= c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x)).$$

Ex: $x^2 y'' - 3xy' + 4y = 0$

$a = 1, \; b = -3, \; c = 4$

Char poly: $r^2 - 4r + 4 = 0$

$\iff$ Double roots $r = 2$

$y = c_1 x^2 + c_2 x^2 (\ln x)$

Ex: $\boxed{x^2y'' - xy' + y = 2x}$

Non-homogeneous

⊕ Homogeneous associated eq is Euler-Cauchy

$$x^2y'' - xy' + y = 0$$

By using the method above you find

$$y_c = c_1y_1 + c_2y_2$$

⊕ Then using y_1, y_2, and $g(x) = 2x$, can find a particular solution y_p (method of undetermined coeff, or variation of parameters)

⊕ Finally $y = y_c + y_p$

<u>Step 1</u>: $x^2y'' - xy' + y = 0$

$a = 1, \ b = -1, \ c = 1.$

Char poly: $r^2 - 2r + 1 = 0 \Rightarrow$ double roots $1, 1$

$$y_c = c_1x^1 + c_2x^1\ln x = c_1x + c_2x\ln x$$

$$y_1 = x, \quad y_2 = x\ln x.$$

<u>Step 2</u>: Find y_p (Variation of parameters)

$$y'' - \frac{1}{x}y' + \frac{y}{x^2} = \frac{2}{x}$$

$$W = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \begin{pmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{pmatrix} = x$$

$$W_1 = \begin{pmatrix} 0 & x\ln x \\ \frac{2}{x} & \ln x + 1 \end{pmatrix} = -2\ln x$$

$$W_2 = \begin{pmatrix} y_1 & 0 \\ y_1' & \frac{2}{x} \end{pmatrix} = \begin{pmatrix} x & 0 \\ 1 & 2/x \end{pmatrix} = 2.$$

$$u = \int \frac{W_1}{W} = -\int \frac{2\ln x}{x}\,dx \quad \left(\text{sub } u = \ln x,\ du = \frac{1}{x}\,dx\right)$$

$$= -(\ln x)^2$$

$$v = \int \frac{W_2}{W} = \int \frac{2}{x} = 2\ln x$$

$$y_p = u y_1 + v y_2 = -(\ln x)^2 x + 2\ln x\; x(\ln x)$$

$$= x(\ln x)^2.$$

<u>Step3</u>: $\quad y = y_c + y_p = c_1 x + c_2(\ln x)x + x(\ln x)^2$